Fast and High Accuracy Numerical Methods for the Solution of Nonlinear Klein–Gordon Equations

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In this work we propose fast and high accuracy numerical methods for the solution of the one-dimensional nonlinear Klein–Gordon (KG) equations. These methods are based on applying fourth-order time-stepping schemes in combination with discrete Fourier transform to numerically solve the KG equations. After transforming each equation to a system of ordinary differential equations, the linear operator is not diagonal, but we can implement the methods such as for the diagonal case which reduces the time in the central processing unit (CPU). In addition, the conservation of energy in KG equations is investigated. Numerical results obtained from solving several problems possessing periodic, single, and breather-soliton waves show the high efficiency and accuracy of the mentioned methods.

Key words: Klein–Gordon Equation; Exponential Time Differencing; Integrating Factor; Spectral Methods; High Accuracy; Soliton; Conservation of Energy.

1. Introduction

The Klein–Gordon (KG) equation, that is also known as Klein–Gordon–Fock equation, arises in the study of theoretical physics [1]. This equation is the relativistic version of the Schrödinger equation. It represents the equation of motion of a quantum scalar or a pseudo-scalar field, which is a field whose quanta are spinless particles. Such a problem appears naturally in the study of some nonlinear dynamical problems of mathematical physics, among them radiation theory, general relativity of scattering, and stability of kinks, vortices, and other coherent structures. The KG equation is known as one of the nonlinear wave equations arising in relativistic quantum mechanics. This equation has attracted much attention in studying solitons and condensed matter physics [2], in investigating the interaction of solitons in collisionless plasma, the recurrence of initial states, in lattice dynamics, and in examining the nonlinear wave equations [1]. The KG equation plays a significant role in many scientific applications such as solid state physics and nonlinear optics theory [3]. The nonlinear KG equation has the general form

$$\frac{\partial^2 u}{\partial t^2}(x,t) - q\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\mathrm{d}V(u(x,t))}{\mathrm{d}u}, \quad (x,t) \in [a,b]\times[0,T], \tag{1}$$

where $\frac{\mathrm{d}V(u(x,t))}{\mathrm{d}u}$ is a nonlinear function of u chosen as the derivative of a potential energy $V(u)$. Equation (1) occurs in a series of physical situations, as the propagation of waves in ferromagnetic materials carrying rotations of the direction of magnetization and of laser pulses in two-state media [4, 7].

The nonlinear KG equations which will be examined in this paper have the following forms [5, 6, 8]:

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\frac{\partial^2 u}{\partial x^2} + \beta u - \gamma u^2 = 0, \tag{2}$$

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha u - \beta u^3 = 0, \tag{3}$$

$$\frac{\partial^2 u}{\partial t^2} - \alpha\frac{\partial^2 u}{\partial x^2} + \beta u - \gamma u^7 = 0, \tag{4}$$

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\frac{\partial^2 u}{\partial x^2} - \sin(u) = 0, \tag{5}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \varphi_1(x), \qquad x \in [a,b], \\ \frac{\partial u}{\partial t}(x,0) &= \varphi_2(x), \quad x \in [a,b], \end{aligned} \tag{6}$$

and the periodic boundary condition

$$u(a,t) = u(b,t), \ \ t \in [0,T]. \tag{7}$$

The main property of (1) is the conservation of energy. The energy E for (1) is given by the following expression [5, 9]:

$$E = E(t) = \frac{1}{2}\int_{\mathbb{R}} \left[(u_t)^2 + q(u_x)^2 - 2V(u)\right] \mathrm{d}x. \tag{8}$$

For (2)–(5), $V(u)$ and q are

$$\begin{aligned} V(u) &= -\frac{\beta}{2}u^2 + \frac{\gamma}{3}u^3, \quad q = \alpha^2, \\ V(u) &= -\frac{\alpha}{2}u^2 + \frac{\beta}{4}u^4, \quad q = \alpha^2, \\ V(u) &= -\frac{\beta}{2}u^2 + \frac{\gamma}{8}u^8, \quad q = \alpha, \\ V(u) &= 1 - \cos(u), \qquad q = \alpha^2, \end{aligned}$$

respectively. In the literature several numerical schemes have been developed for solving KG equations. Strauss and Vázquez [10] derived a three-time level scheme with conserved energy using standard finite-difference approximations. Li and Vu-Quoc [11] studied the finite difference invariant structure of a class of algorithms for the nonlinear KG equation and derived algorithms that preserve energy or linear momentum. Jiménez and Vázquez [9] analysised four finite difference schemes for approximating the nonlinear KG equation. They observed undesirable characteristics in some of the numerical schemes, in particular a loss of spatial symmetry and the onset of instability for large values of a parameter in the initial condition of the equation. In [12], an analysis of the schemes described in [9] as applied to a linear problem is carried out, and these indicate that the instability arises from the use of explicit finite difference schemes rather than any failure of energy conservation. This conjecture is further supported by an analysis of two further schemes. The KG equation is solved in [13] using the variational iteration method. Guo et al. developed a conservative Legendre spectral method in [14]. The author of [15] obtained the approximate and/or exact solutions of the generalized Klein–Gordon and sine-Gordon-type equations. With the aid of the symbolic computation system Mathematica, many exact solutions for the KG equation with a quadratic nonlinearity are constructed in [16]. Abbasbandy in [17] presented a numerical solution of nonlinear KG equations with power law nonlinearities by the application of He's variational iteration method. Dehghan and Shokri in [18, 19] proposed a numerical method based on radial bases functions. Also the boundary integral equation approach for solving the one-dimensional sine-Gordon equation (5) is proposed in [20]. A numerical method based on employing the boundary integral equation method and the dual reciprocity boundary element method (DRBEM) is suggested in [21]. Some compact finite difference approaches for the solution of KG problems are given in [22–24]. A spline collocation approach for the solution of the KG equation is presented in [25]. The method of lines approach is used in [5] to transform the sine-Gordon equation into a first-order nonlinear initial-value problem and then replacing the matrix exponential term in a recurrence relation by rational approximation which leads to the second-order methods in both space and time variables. Bratsos proposed another approach in [6] for solving (5) which has second-order accuracy in space and fourth-order accuracy in the time variable. Finally, Bratsos in [4, 7] developed a predictor-corrector (PC) scheme based on the use of rational approximation of second order to the matrix exponential term in a three-time level recurrence relation.

Most of the existing methods in the literature for solving KG equations are time consuming schemes and have a low order of accuracy. In this paper we propose some numerical schemes for solving (2)–(5) with periodic boundary conditions which are fast and accurate. These methods are based on applying fourth-order time-stepping schemes in combination with discrete Fourier transform. The outline of this paper is as follows. In Section 2, we state the spatial discretization and implementation of the methods and give an approach to save the linear operator of the problems as diagonal case. In Section 3, we briefly introduce the exponential integrators schemes such as the Runge–Kutta integrating factor (IFRK), the Runge–Kutta exponential time differencing (ETDRK) methods, and the Cauchy integral approach of Kassam and Trefethen [26] for calculating ETDRK coefficients. In Section 4, we report the numerical experiments of solv-

ing KG equations with the applied method for several problems, and the conservation of energy is presented. Finally, a conclusion is drawn in Section 5.

2. Spatial Discretization

The spatial discretisation for (2)–(5) is done using a Fourier spectral method with periodic boundary conditions [27, 28]. It is given a function u which is periodic on an appropriate spatial grid x_j. From the definition of discrete Fourier transform (DFT), we have [28]

$$\widehat{u}_k = h\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}kx_j}u_j,\ \ k=-\frac{N}{2}+1,...,\frac{N}{2},$$

in which N is the number of grid points on a periodic grid, h is the spacing of the grid points, and k are the Fourier wave numbers. The inverse DFT is

$$u_j = \frac{1}{2\pi}\sum_{k=-N/2+1}^{N/2} \mathrm{e}^{\mathrm{i}kx_j}\widehat{u}_k,\ \ j=1,\ldots,N.$$

Let w be the nth derivative of v. For calculating w, we first compute $\widehat{v}$ then put $\widehat{w}=(\mathrm{i}k)^n\widehat{v}$. We can obtain w by applying the inverse Fourier transform.

If we show the general form of (2)–(5) with $u_{tt}=\alpha^2u_{xx}+F(u,t)$ and put $u_t=v$ then the following system of partial differential equations (PDEs) is resulted:

$$\begin{aligned} u_t &= v\\ v_t &= \alpha^2u_{xx}+F(u,t). \end{aligned} \tag{9}$$

If we show $U=[u\ \ v]^{\mathrm{T}}$ by applying the DFT method to (9) and leaving the time component t, the following system of ordinary differential equations (ODEs) is obtained:

$$\widehat{U}_t = L\widehat{U}+\widehat{N(U)}, \tag{10}$$

where $\widehat{N(U)}=[\widehat{v}\ \ \widehat{F(u,t)}]^{\mathrm{T}}$ and the linear operator L has the following non-diagonal form:

$$L_{2N\times 2N}=\begin{bmatrix} 0 & 0\\ D_{N\times N} & 0\end{bmatrix}, \tag{11}$$

where $D_{N\times N}$ is a diagonal matrix whose diagonal entries are $-\alpha^2k^2$. In applying DFT in combination with exponential integrators, we need only to store D and implement the methods as for the diagonal case. In fact, in the exponential integrators we need to calculate the inverse and exponential of the matrix ΔtL which have definite structures and are stated in the following lemmas.

Lemma 2.1 *The exponential of matrix ΔtL in (11) is:*

$$\mathrm{e}^{\Delta tL}=\begin{bmatrix} I_{N\times N} & 0\\ \Delta tD & I_{N\times N}\end{bmatrix},$$

where $I_{N\times N}$ is the identity matrix of size N.

Proof. It is easy to check that $L^i=0,\ \ i=2,3,\ldots$ So from Taylor expansion we have $\mathrm{e}^{\Delta tL}=I_{2N\times 2N}+\Delta tL$. □

We only store the vector $q=[q_1\ q_2]^{\mathrm{T}}$, where $q_1=[1,\ldots,1]$ and $q_2=-\Delta t\alpha^2k^2$, and do all computations of the method on this vector. The next lemma is used in ETDRK and ETDRKB methods.

Lemma 2.2 *The inverse of matrix $zI-\Delta tL$, $z\neq 0$, $z\in\Gamma$ is:*

$$(zI-\Delta tL)^{-1}=\frac{1}{z^2}(zI+\Delta tL).$$

Proof. It is clear that $zI-\Delta tL$, $z\neq 0$, is invertible and $(zI-\Delta tL)^{-1}(zI-\Delta tL)=I$. □

Also in this case we store vector $q=[q_1\ q_2]^{\mathrm{T}}$, where $q_1=\frac{1}{z}[1,\ldots,1]$ and $q_2=\frac{1}{z^2}\Delta t(-\alpha^2k^2)$, and do all computations of the method on this vector.

3. Exponential Integrators

Exponential integrators are numerical schemes specifically designed for solving differential equations where it is possible to discretized the original PDE into a linear and a nonlinear part and obtain a coupled system of ODEs,

$$u_t = Lu+N(u,t). \tag{12}$$

In (12) Lu is the linear part and $N(u,t)$ is the nonlinear part. The aim of the exponential integrators is to treat the linear term exactly and allow the remaining part of the integration to be integrated numerically using an explicit scheme. In this paper we implement exponential integrators of the Runge–Kutta type. We consider the Runge–Kutta integrating factor (IFRK) [26, 29, 30], the Runge–Kutta exponential time differencing (ETDRK) [26, 29], and the ETDRK method with improved accuracy by Krogstad (ETDRKB) [31]. Further, we will use the numerically stable scheme by Kassam and Trefethen [26] for calculating the coefficients in the ETDRK methods. We briefly introduce these methods.

3.1. Runge–Kutta Integrating Factor (IFRK)

The idea is to make a change of variable that allows us to solve the linear part exactly and then use a numerical scheme of our choice to solve the transformed nonlinear equation. Starting with our discretised PDE (12), we define

$$v = \mathrm{e}^{-Lt}u. \tag{13}$$

Differentiating (13) gives

$$v_t = -\mathrm{e}^{-Lt}Lu + \mathrm{e}^{-Lt}u_t. \tag{14}$$

If we multiply (12) by the integrating factor e^{-Lt}, we have

$$\mathrm{e}^{-Lt}u_t - \mathrm{e}^{-Lt}Lu = \mathrm{e}^{-Lt}N(u), \tag{15}$$

which gives

$$v_t = \mathrm{e}^{-Lt}N(\mathrm{e}^{Lt}\,v). \tag{16}$$

Now we can use a time-stepping method of our choice to advance the transformed equation. We use a fourth-order Runge–Kutta formula and obtain the IFRK scheme. Regarding to (10), which is in the Fourier space, the fourth-order formula of the IFRK method to solve (12) is as follows [26, 29, 30]:

$$\begin{aligned}
&A = \Delta t\mathcal{R}(\mathcal{F}(N(\mathcal{F}^{-1}(u)))),\\
&B = \Delta t\mathcal{R}(\mathcal{F}(N(\mathcal{F}^{-1}(\mathrm{e}^{\Delta tL/2}(u+A/2))))),\\
&C = \Delta t\mathcal{R}(\mathcal{F}(N(\mathcal{F}^{-1}(\mathrm{e}^{\Delta tL/2}u+B/2)))),\\
&D = \Delta t\mathcal{R}(\mathcal{F}(N(\mathcal{F}^{-1}(\mathrm{e}^{\Delta tL}u+\mathrm{e}^{\Delta tL/2}C)))),\\
&u_{n+1} = \mathrm{e}^{\Delta tL}u_n + \frac{1}{6}(\mathrm{e}^{\Delta tL}A + 2\mathrm{e}^{\Delta tL/2}(B+C)+D),
\end{aligned} \tag{17}$$

where $\mathcal{R}(.)$, $\mathcal{F}(.)$, and $\mathcal{F}^{-1}(.)$ show the real part, the Fourier transform, and the inverse Fourier transform of considered functions, respectively.

3.2. Runge–Kutta Exponential Time Differencing (ETDRK)

The idea of the ETD methods is similar to the method of the integrating factor. We multiply both sides of a differential equation by some integrating factor, then we make a change of variable that allows us to solve the linear part exactly. In the derivation of the ETD methods, instead of making a complete change of variable, we integrate (15) over a single time step of length Δt (from $t = t_n$ to $t = t_{n+1} = t_n + \Delta t$), getting

$$\begin{aligned}
u_{n+1} = {}& \mathrm{e}^{\Delta tL}u_n\\
&+ \mathrm{e}^{\Delta tL}\int_0^{\Delta t}\mathrm{e}^{-\Delta tL}N(u(t_n+\tau),t_n+\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{18}$$

The various ETD methods come from how one approximates the integral in this expression. Cox and Matthews derived in [29] a set of ETD methods based on the Runge–Kutta time stepping, which they called ETDRK methods. The fourth-order ETDRK scheme formula is as follows [29]:

$$\begin{aligned}
u_{n+1} = {}& u_n\mathrm{e}^{L\Delta t} + \alpha N(u_n,t_n) + 2\beta[N(a_n,t_n+\Delta t/2)\\
&+ N(b_n,t_n+\Delta t/2)] + \gamma N(c_n,t_n+\Delta t),\\
\alpha = {}& \Delta t^{-2}L^{-3}[-4-\Delta tL+\mathrm{e}^{\Delta tL}(4-3\Delta tL+(\Delta tL)^2)],\\
\beta = {}& \Delta t^{-2}L^{-3}[2+\Delta tL+\mathrm{e}^{\Delta tL}(-2+\Delta tL)],\\
\gamma = {}& \Delta t^{-2}L^{-3}[-4-3\Delta tL-(\Delta tL)^2+\mathrm{e}^{\Delta tL}(4-\Delta tL)],\\
a_n = {}& \mathrm{e}^{\Delta tL/2}u_n + L^{-1}(\mathrm{e}^{\Delta tL/2}-I)N(u_n,t_n),\\
b_n = {}& \mathrm{e}^{\Delta tL/2}u_n + L^{-1}(\mathrm{e}^{\Delta tL/2}-I)N(a_n,t_n+\Delta t/2),\\
c_n = {}& \mathrm{e}^{\Delta tL/2}a_n + L^{-1}(\mathrm{e}^{\Delta tL/2}-I)(2N(b_n,t_n+\Delta t/2)\\
&- N(u_n,t_n)).
\end{aligned} \tag{19}$$

It is shown in [31] that the main step of Cox–Matthews method can be reproduced based on the techniques of continuous Runge–Kutta methods. Motivated from the same idea, but also applied to the internal stages of the method, a new fourth-order method is derived in [31]. This method, which is also based on the classical fourth-order Runge–Kutta method is as follows:

$$\begin{aligned}
u_{n+1} = {}& \mathrm{e}^{\Delta tL}u_n + \Delta t[4\phi_2(\Delta tL) - 3\phi_1(\Delta tL)\\
&+ \phi_0(\Delta tL)]N(u_n,t_n)\\
&+ 2\Delta t[\phi_1(\Delta tL) - 2\phi_2(\Delta tL)]N(a_n,t_n+\Delta t/2)\\
&+ 2\Delta t[\phi_1(\Delta tL) - 2\phi_2(\Delta tL)]N(b_n,t_n+\Delta t/2)\\
&+ \Delta t[4\phi_2(\Delta tL) - \phi_1(\Delta tL)]N(c_n,t_n+\Delta t/2),
\end{aligned}$$

where

$$\begin{aligned}
a_n = {}& \mathrm{e}^{\Delta tL/2}u_n + \Delta t/2\phi_0(\Delta tL/2)N(u_n,t_n),\\
b_n = {}& \mathrm{e}^{\Delta tL/2}u_n + \Delta t/2[\phi_0(\Delta tL/2) - 2\phi_1(\Delta tL/2)]\\
&\cdot N(u_n,t_n) + \Delta t\phi_1(\Delta tL/2)N(a_n,t_n+\Delta t/2),\\
c_n = {}& \mathrm{e}^{\Delta tL}u_n + \Delta t[\phi_0(\Delta tL) - 2\phi_1(\Delta tL)]N(u_n,t_n)\\
&+ 2\Delta t\phi_1(\Delta tL)N(b_n,t_n+\Delta t).
\end{aligned}$$

Also the functions ϕ_i are defined as

$$\phi_0(z) = \frac{e^z - 1}{z}, \quad \phi_1(z) = \frac{e^z - 1 - z}{z^2},$$
$$\phi_2(z) = \frac{e^z - 1 - z - z^2/2}{z^3}.$$

We label this fourth-order time-stepping method as ETDRKB scheme to distinguish it from a standard Runge–Kutta scheme and to be consistent with the notation of [26, 29, 32].

Unfortunately, the ETDRK schemes suffer from numerical instability when the linear operator L has eigenvalues close to zero, because disastrous cancellation errors arise. Kassam and Trefethen in [26] have studied these instabilities and have found that they can be removed by evaluating a certain integral on a contour that is separated from zero. The procedure is basically to change the evaluation of the coefficients, which is mathematically equivalent to the original ETDRK scheme of [29], but in [33] it has been shown to have the effect of improving the stability of integration in time. Also, it easily can be implemented and the impact on the total computing time is small.

The approach of [26] is to evaluate $f(z)$ via an integral over a contour in the complex plane that encloses z and is well separated from 0 and is

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z}\, dt,$$

where the contour Γ must contain z in its interior and $i^2 = -1$. This formula is the well known Cauchy integral formula. It transforms our problem to one of evaluating our function over a contour well away from the problem area. For matrices, a similar form exists [26], i.e.

$$f(L) = \frac{1}{2\pi i} \int_\Gamma (tI - L)^{-1} f(t)\, dt,$$

in which Γ is any contour that encloses the eigenvalues of L. Contour integrals of analytic functions in the complex plane are easy to evaluate by means of the trapezoid rule.

4. Numerical Experiments

To study the validity and effectiveness of these methods and compare the accuracy of the proposed numerical schemes with other techniques known in the bibliography, they are applied to various problems. We performed our computations using Matlab 7 software on a Pentium IV, 2000 MHz CPU machine with 2 Gbyte of memory. The integration in (8) was performed using the composite trapezoidal rule. In all problems we use a 512-point Fourier spectral discretization in x. Also we use fast Fourier transform (FFT) routines in Matlab (i.e. fft and ifft) to calculate Fourier transform and inverse Fourier transform.

4.1. Problem 1

Consider the partial differential equation (3),

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} + \alpha u - \beta u^3 = 0.$$

We solve this PDE with two different initial conditions.

4.1.1. Periodic Waves

We consider (3) with $\alpha = 1$ and $\beta = 0.1$ on the region $0 < x < 1.28$ and the initial conditions

$$\varphi_1(x) = A\left[1 + \cos\left(\frac{2\pi x}{1.28}\right)\right],$$
$$\varphi_2(x) = 0.$$

For the above problem, and due to the periodic boundary conditions, the continuous solutions remain always symmetric with respect to the center of the spatial interval [4, 9]. Authors of [9] studied this problem and found undesirable characteristics in some of the numerical schemes, in particular a loss of spatial symmetry and the onset of instability for larger values of the parameter A (amplitude) in the initial condition of the equation. Also, it was found that the numerical results given in [4] were more accurate than the other results given in [9, 10, 18]. In Figure 1, we show the approximate solutions of Problem 4.1 with $A = 1.5$ and $A = 150$. As we see, the calculated approximate solutions are similar to the results of [4]. Also the approximate solutions obtained remain symmetric with respect to the center of the spatial interval, and the solution remained bounded for amplitude $A = 150$ when $t \in [0, 36]$. Table 1 gives the energy $E(t)$ at various time levels that show that the energy is conserved.

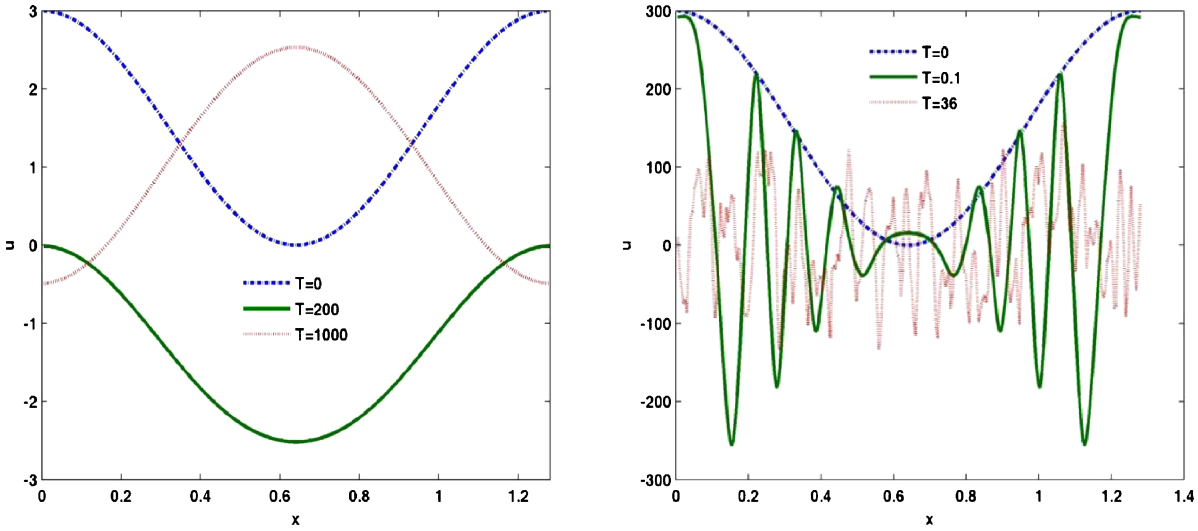

Fig. 1 (colour online). Approximate solutions of the periodic wave problem with $A = 1.5$ at $T = 0$, 200, 1000 (left panel) and with $A = 150$ at $T = 0$, 0.1, 36 (right panel).

Table 1. Errors of calculated energy for periodic wave with the IFRK method when $t \in [0, 1000]$ and $A = 1.5$, $\Delta t = 0.001$, $N = 250$, $E(0) = 26.59641398628989$.

Time (t)	$t = 1$	$t = 100$	$t = 500$	$t = 1000$
$\lvert E(t) - E(0)\rvert$	1.8×10^{-10}	9.8×10^{-10}	4.6×10^{-9}	8.9×10^{-9}

4.1.2. Single Soliton

We consider the partial differential equation (3) with the following exact solution:

$$u(x,t) = \sqrt{\frac{2\alpha}{\beta}}\ \mathrm{sech}\,(\lambda(x-ct)),\ \ -10 \le x \le 10,$$

where $\lambda = \sqrt{\frac{\alpha}{\alpha^2-c^2}}$, α, β, $\alpha^2 - c^2 > 0$. The initial conditions can be obtained from the exact solution. The exact solution represents a soliton which travels with velocity c and whose amplitude is governed by the real parameter $\sqrt{\frac{2\alpha}{\beta}}$. This problem is given in [4]. For comparison, we put parameters α, β, and c similar to [4], i.e. $\alpha = 0.3$, $\beta = 1$, and $c = 0.25$. We solved the above problem with the methods presented in this article for several values of Δt and $-10 \le x \le 10$ at final time $T = 10$. Figure 2 shows the convergence of the applied methods for this problem. As we see, the methods achieve an accuracy of order 10^{-11} in 10 s. The IFRK method achieves better results in comparison with ETD and ETDRK methods. In Table 2, the conservation of energy is shown.

4.2. Problem 2

We consider the partial differential equation (2)

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta u - \gamma u^2 = 0,\ \ -30 \le x \le 30,$$

with the following exact solution

$$u(x,t) = \frac{3\beta}{2\gamma}\ \mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\beta}{\alpha^2-c^2}}(x-ct)\right).$$

The initial conditions can be obtained from the exact solution. This problem is used in [8]. We solved the above problem with the methods presented in this article for several values of Δt and $-30 \le x \le 30$ at final time $T = 10$. We put $\alpha = 0.3$, $\beta = 0.1$, $\gamma = 1$, and $c = 0.1$. Figure 3 shows the convergence of the applied methods for this problem. As we see, the methods achieve an accuracy of order about 10^{-14} in 10 s. Also from this figure, we can conclude that for this problem all of the methods have similar results. Table 3 gives the energy $E(t)$ at various time levels that show that the energy is conserved.

4.3. Problem 3

We consider the partial differential equation (4),

$$\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta u - \gamma u^7 = 0,\ \ -10 \le x \le 10,$$

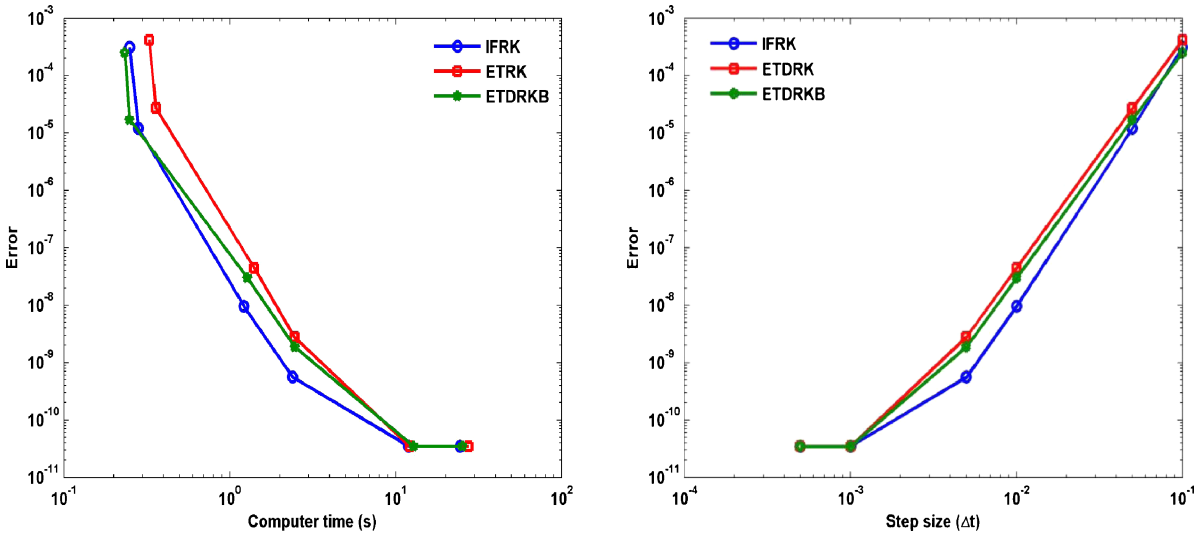

Fig. 2 (colour online). Convergence of applied methods for single soliton problem for $-10 \leq x \leq 10$ at final time $T = 10$. The methods achieve an accuracy of order about 10^{-11} in 10 s.

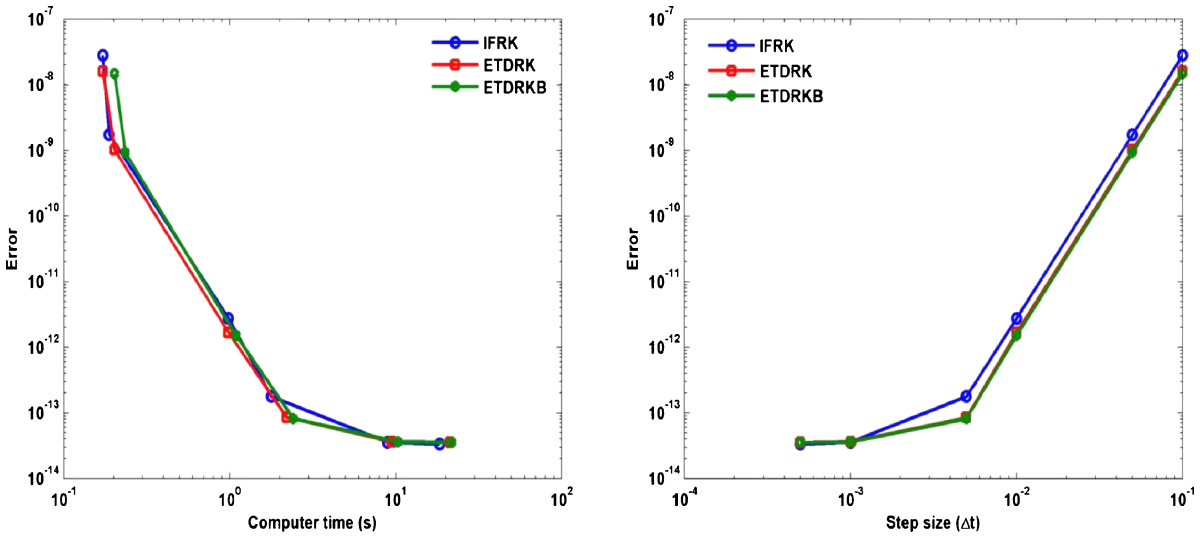

Fig. 3 (colour online). Convergence of applied methods for Problem 2 for $-30 \leq x \leq 30$ at final time $T = 10$. The methods achieve an accuracy of order about 10^{-14} in 10 s.

Table 2. Errors of calculated energy for single soliton with the IFRK method when $t \in [0, 20]$ and $\Delta t = 0.01$, $N = 250$, $E(0) = 0.11890408663357$.

Time (t)	$t = 1$	$t = 5$	$t = 10$	$t = 20$
$\lvert E(t) - E(0)\rvert$	3.5×10^{-12}	1.3×10^{-11}	2.6×10^{-11}	5.2×10^{-11}

Table 3. Errors of calculated energy for Problem 2 with the IFRK method when $t \in [0, 20]$ and $\Delta t = 0.01$, $N = 250$, $E(0) = 0.00314710595919$.

Time (t)	$t = 1$	$t = 5$	$t = 10$	$t = 20$
$\lvert E(t) - E(0)\rvert$	1.2×10^{-16}	3.1×10^{-16}	9.2×10^{-16}	3.0×10^{-14}

Table 4. Errors of calculated energy for Problem 3 with the IFRK method when $t \in [0, 20]$ and $\Delta t = 0.001$, $N = 512$, $E(0) = 0.13997199493915$.

Time (t)	$t = 1$	$t = 5$	$t = 10$	$t = 20$
$\lvert E(t) - E(0)\rvert$	1.1×10^{-12}	8.8×10^{-12}	1.2×10^{-9}	1.7×10^{-5}

Table 5. Errors of calculated energy for Problem 4 with the IFRK method when $t \in [0, 20]$ and $\Delta t = 0.001$, $N = 512$, $E(0) = 14.31083505599865$.

Time (t)	$t = 5$	$t = 10$	$t = 20$
$\lvert E(t) - E(0)\rvert$	1.0×10^{-14}	1.1×10^{-14}	1.1×10^{-14}

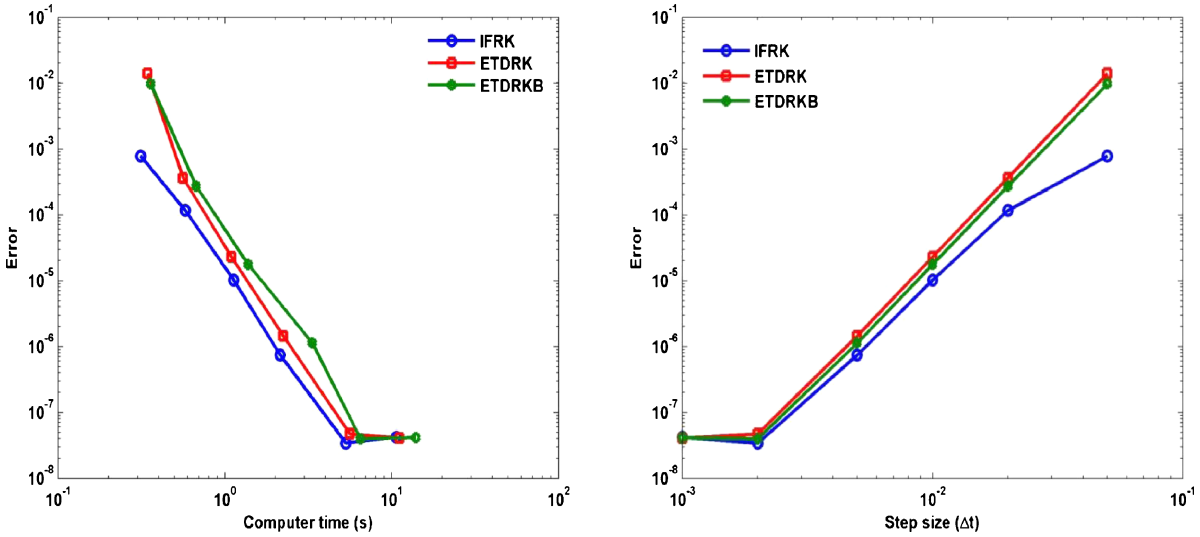

Fig. 4 (colour online). Convergence of applied methods for Problem 3 for $-10 \leq x \leq 10$ at final time $T = 10$. The methods achieve an accuracy of order about 10^{-8} in under 10 s.

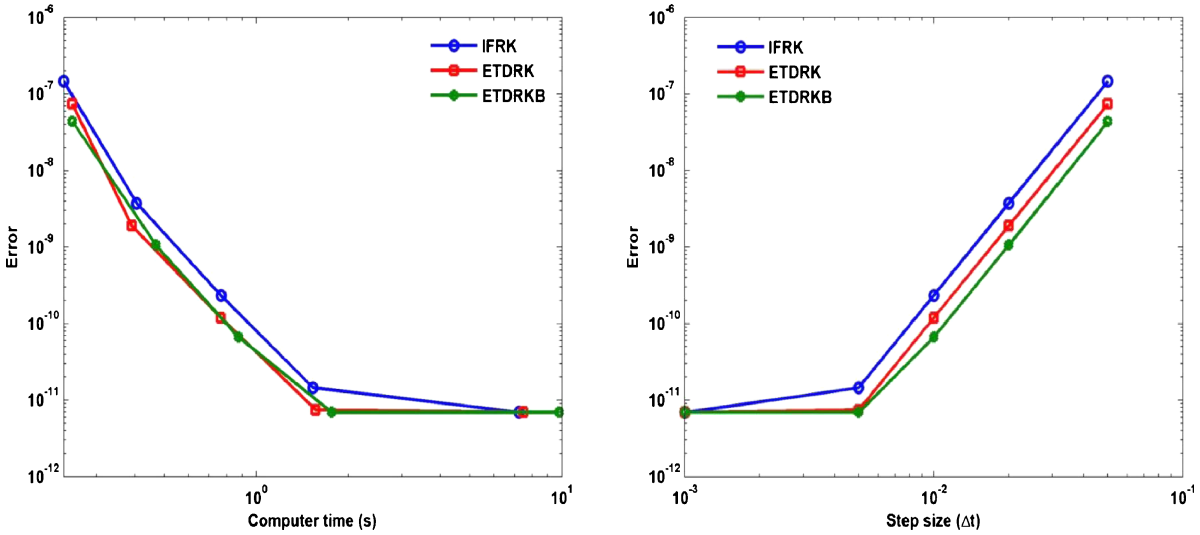

Fig. 5 (colour online). Convergence of applied methods for Problem 4 for $-30 \leq x \leq 30$ at final time $T = 10$. The methods achieve an accuracy of order about 10^{-11} in under 10 s.

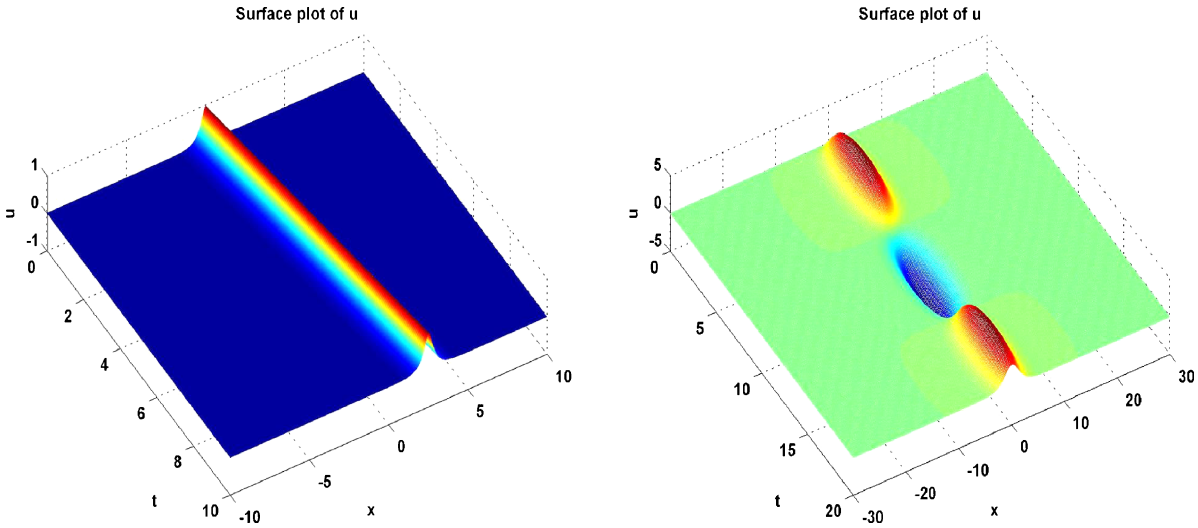

Fig. 6 (colour online). Surface plots of approximate solutions for Problems 3 (left panel) and 4 (right panel).

with the following exact solution:

$$u(x,t) = \sqrt[6]{\frac{4b\,\mathrm{sech}^2\left(3\sqrt{\frac{b}{a-c^2}}(x-ct)\right)}{k}}.$$

The initial conditions can be obtained from the exact solution. This problem is used in [3, 8]. We solved the above problem for several values of Δt and $-10 \le x \le 10$ at final time $T = 10$. Figure 4 shows the convergence of the applied methods for this problem and in Table 4, the conservation of energy is shown. From this figure, we can conclude that the IFRK method achieves better results in comparison with ETD and ETDRK methods.

4.4. Problem 4: Breather Soliton

We consider the partial differential equation (5),

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} - \sin(u) = 0, \quad -30 \le x \le 30,$$

with the following exact solution:

$$u(x,t) = 4\tan^{-1}\left(\frac{\sin(c\lambda t)\,\mathrm{sech}(x\lambda)}{c}\right),$$

where $\lambda = \frac{1}{\sqrt{1+c^2}}$. The initial conditions can be obtained from the exact solution. This problem is given in [4, 20], known as the breather solution of the sine-Gordon equation, and represents a pulse-type structure of a soliton. We solved this problem with the methods presented in this paper for several values of Δt and $-30 \le x \le 30$ at final time $T = 10$. Figure 5 shows the convergence of the applied methods for this problem. As we see, the methods achieve an accuracy of order 10^{-11} in under 10 seconds and for this problem the ETDRK method achieves better results in comparison with IFRK and ETD methods. Table 5 gives the energy $E(t)$ at various time levels that show that the energy is conserved. Figure 6 presents the surface plots of approximate solutions for Problems 3 and 4.

Most of the existing methods in the literature for the numerical solution of the KG equation are time consuming and have low order of accuracy. From Tables 1–5 and Figures 1–5, we can conclude that the proposed methods are fast and have a high order of accuracy.

5. Concluding Remarks

We have applied fourth-order time-stepping schemes (IFRK, ETDRK, and ETDRKB) in combination with discrete Fourier transform to numerically solve KG equations with periodic boundary conditions and achieved excellent results (in both accuracy and CPU time). After transforming the equations

to a system of ordinary differential equations, the linear operator is not diagonal, but we can implement the methods such as for the diagonal case and reduce the CPU time. For all problems the conservation of energy was investigated and the corresponding tables were presented. It would be interesting to implement these methods for non-periodic boundary conditions and two-dimensional Klein–Gordon problems which is the subject of our future work.

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